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A BOUNDARY DISCONTINUOUS FOURIER SOLUTION FOR CLAMPED TRANSVERSELY ISOTROPIC (PYROLYTIC GRAPHITE) MINDLIN PLATES

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Abstract—An analytical solution to the long-standing boundary-value problem of a shear-flexible (moderately-thick) rigidly clamped transversely isotropic rectangular plate, subjected to transverse loading, is presented. A recently developed accurate yet computationally efficient boundary discontinuous Fourier series technique (BDFST) has been utilized to solve the three highly coupled second-order partial differential equations with constant coefficients that result from the Mindlin hypothesis. Numerical results presented (i) testify to the accuracy and computational efficiency of the above-mentioned solution methodology, (ii) help in understanding the nature of convergence of double Fourier series in the presence of edge discontinuities introduced by the fully clamped boundary conditions, (iii) ascertain the limit of applicability of the classical plate theory (CPT), and (iv) provide physical insight into such complex deformation behaviors as the effect of transverse shear deformability and thickness on the deformation of rigidly clamped moderately-thick plates of metallic (isotropic) and pyrolytic graphite (transversely isotropic) constructions.

1. INTRODUCTION

An ever-expanding quest for novel materials, capable of maintaining structural integrity in the most severe thermo-mechanical environments, has lured the aerospace and missile industry into the development of refractory materials, such as pyrolytic graphite [see Brunelle (1971) and Zukas and Vinson (1971)]. A pyrolytic graphite plate is transversely isotropic, with the ratio of in-plane Young's modulus to transverse shear modulus, ranging between 20 and 50, rendering pyrolytic graphite among the most transversely shear deformable materials known to man. Any realistic analysis of such plates must, therefore, include the effect of transverse shear deformation, even for sufficiently thin geometry.

Analytical or strong forms of solutions in the form of Fourier series to the problems of thin homogeneous isotropic rectangular plates, characterized by the classical plate theory (CPT) or the Kirchhoff hypothesis, are well documented in the literature [see, e.g. Timoshenko and Woinowsky-Krieger (1959) and Szilard (1974)], because of their utility in the design of aircraft, spacecraft, marine and ground-based structures of metallic construction. Navier's method, which is the most straightforward, requires the assumed double Fourier series-based solution functions to satisfy both the governing partial differential equation (PDE) and appropriate (geometric and natural) boundary conditions *a priori*, and as a result, is restricted to simply-supported boundary conditions, prescribed at all four edges. Levy's method, which may be regarded as an improvement over Navier's method, relies on the use of ordinary (single) Fourier series, for solving a class of problems of rectangular isotropic plates requiring two simply-supported opposite edges, while the other two edges may be clamped, simply supported, free or any of their combinations, thereby reducing the governing PDE to an ordinary differential equation (ODE). Navier's and Levy's approaches are, therefore, of limited utility, since they both require satisfaction of governing PDEs and simply-supported boundary conditions *a priori* and are not applicable to other types of boundary conditions (e.g. all edges rigidly clamped) in a direct manner.

Timoshenko and Woinowsky-Krieger (1959) have analytically solved the problem of a

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fully clamped thin (CPT-based) rectangular isotropic plate in an indirect manner, utilizing the method of Levy, in conjunction with a superposition technique. Green (1944) has taken a more direct approach and has obtained double Fourier series solutions to the problems of bending and stability of thin (CPT-based) fully clamped isotropic plates, where he has solved one fourth-order PDE in conjunction with appropriate geometric boundary conditions. The approach adopted by Green (1944), originally expounded by Hobson (1926) and recently generalized by Chaudhuri (1989), is an extension of Goldstein's (1936, 1937) work, involving ordinary Fourier series, on the stability of fluid flow and will be referred to as belonging to the class of boundary discontinuous Fourier series technique (BDFST), wherein dicontinuities in the solution functions and/or their derivatives are accounted for. Whitney (1971) has further extended the work of Green (1944) to solve the problems of vibration and stability of fully clamped rectangular anisotropic plates. Unlike Green (1944), Whitney (1970) has resorted to using two sets of double Fourier series functions, although any specific reason or basis for this procedure has not been provided.

An extensive literature search reveals that the available Fourier series solutions to the problems of moderately-thick isotropic and transversely isotropic rectangular plates, typically characterized by the Mindlin (1951) hypothesis, are generally limited to the aforementioned simply-supported boundary conditions, required by Navier's and Levy's approaches [see, e.g. Salerno and Goldberg (1960) and Brunelle (1971)]. Fourier series type analytical solutions to the important problem of a rigidy clamped Mindlin plate appear to be non-existent. A number of approximate weak (integral) forms of solutions to the class of boundary-value problems under investigation are, however, available in the literature. Examples include Ritz–Galerkin approaches with (a) global [see, e.g. Timoshenko and Woinowsky-Krieger (1959) and Szilard (1974)] or (b) local [e.g. finite element methods: Zienkiewicz (1977), Hrabok and Hrudey (1984), Chaudhuri (1987)] supports. It is important to note that strong and weak forms of solution are fundamentally different in that solutions are sought in different function spaces. For the problem under investigation, a strong form of solution will dictate that the solution sought must belong to a space of square integrable functions with square integrable first and second partial derivatives in the interior of the domain, while for a weak form of solution, the restriction of square integrable second partial derivatives need not be imposed [see, for definition Hughes (1987)]. Additionally, the stresses (or moments), computed using the finite element methods (FEM), are accurate only at the Barlow points (by virtue of the mean value theorem), once the displacements at the nodes have reached convergence. Because of these reasons, it is a standard practice to check the acccuracy of the approximate weak forms of solution, such as those computed using the FEM, against their analytical or strong form counterparts (e.g. Fourier series). Chaudhuri's (1987) recent attempt to verify his Mindlin type degenerate triangular plate bending element for the rigidly clamped boundary conditions has met with frustration, because of lack of availability of analytical solutions for this type of boundary condition, which is one of the objectives of the present investigation.

Recently, Chaudhuri (1989) has developed a novel method for obtaining boundary-discontinuous double Fourier series solutions to a system of completely coupled linear PDEs, subjected to completely coupled boundary conditions. This method (i) guides the selection of appropriate assumed double Fourier series solution functions, depending on the coefficients of the system of governing PDEs, (ii) guides in making decisions with regard to the discontinuities either in the assumed solution functions or their first derivatives, (iii) ensures the existence of the solution, depending on the coefficients of the boundary condition equations, and finally (iv) leads to a highly efficient computational scheme in spite of the complexity of the completely (or highly) coupled PDEs. The primary objective of the present study is to apply this technique to obtaining a solution to the three highly coupled second-order PDEs (in contrast to one fourth-order PDE for the case of CPT) with constant coefficients, together with the rigidly clamped (Dirichlet) boundary conditions (prescribed at all four edges) and fill in the aforementioned analytical vacuum. Presentation of numerical results for various parametric ratios, and comparison with the available CPT-based analytical and FSDT (first order shear deformation theory, based on Mindlin hypothesis)-based finite element solutions will comprise the second objective of this investigation.

2. THEORETICAL FORMULATION

Figure 1 shows a plate of thickness h, with the reference axes, x and y, being located at the midsurface, while the z-axis denotes the normal direction. a and b are the dimensions of the plate along the x- and y-axes, respectively. The relations between strain components, ε_x, ε_y, γ_{xy}, γ_{yz}, γ_{xz} and reference surface displacement and rotations at a point (x, y, z) inside a plate are given by

$$\varepsilon_x = z\kappa_x, \quad \varepsilon_y = z\kappa_y, \quad \varepsilon_{xz} = \varepsilon^0_{xz}, \quad \varepsilon_{yz} = \varepsilon^0_{yz}, \quad \varepsilon_{xy} = z\kappa_{xy}, \tag{1}$$

where

$$\varepsilon^0_{xz} = w_{,x}+\phi_x, \quad \varepsilon^0_{yz} = w_{,y}+\phi_y, \quad \kappa_x = \phi_{x,x}, \quad \kappa_y = \phi_{y,y}, \quad \kappa_{xy} = \phi_{x,y}+\phi_{y,x}, \tag{2}$$

in which w, ϕ_x and ϕ_y denote the transverse displacement (deflection) and rotations of the normal to the reference surface about the y- and x-axes, respectively, while κ_x, κ_y and κ_{xy} represent the reference surface curvatures and twist. The equilibrium equations for a plate, subjected to uniformly distributed load, q, are:

$$Q_{x,x}+Q_{y,y}+q = 0, \quad M_{x,x}+M_{xy,y}-Q_x = 0, \quad M_{xy,x}+M_{y,y}-Q_y = 0, \tag{3}$$

wherein the stress couples M_x, M_y, M_{xy} and shear stress resultants, Q_x, Q_y, in terms of the components of displacement and rotation, are given by

$$M_x = D(\phi_{x,x}+\nu\phi_{y,y}), \quad M_y = D(\nu\phi_{x,x}+\phi_{y,y}), \quad M_{xy} = \frac{D(1-\nu)}{2}(\phi_{x,y}+\phi_{y,x}),$$

$$Q_x = A(w_{,x}+\phi_x), \quad Q_y = A(w_{,y}+\phi_y), \tag{4}$$

where, for a transversely isotropic plate, with plate midplane being the plane of isotropy [see Brunelle (1971)],

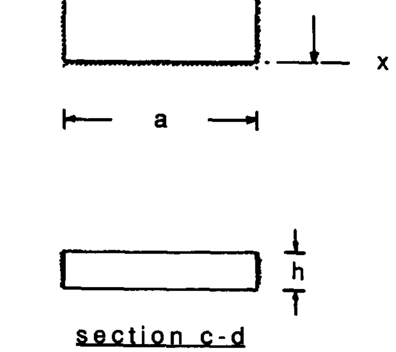

Fig. 1. A rectangular plate.

$$D = Eh^3/\{12(1-\nu^2)\}, \quad A = G_t hk^2, \quad G_t \neq E/\{2(1+\nu), \tag{5}$$

in which E, G_t, ν and k^2 are the in-plane Young's modulus, transverse shear modulus, in-plane Poisson's ratio and shear correction factor, respectively. Introduction of eqns (5) into eqns (4) yields the following three second-order partial differential equations:

$$A(w_{,xx}+w_{,yy}+\phi_{x,x}+\phi_{y,y}) = -q,$$

$$D\left\{\phi_{x,xx}+\left(\frac{1+\nu}{2}\right)\phi_{y,xy}+\left(\frac{1-\nu}{2}\right)\phi_{x,yy}\right\}-A(w_{,x}+\phi_x) = 0,$$

$$D\left\{\phi_{y,yy}+\left(\frac{1+\nu}{2}\right)\phi_{x,xy}+\left(\frac{1-\nu}{2}\right)\phi_{y,xx}\right\}-A(w_{,y}+\phi_y) = 0. \tag{6}$$

The rigidly clamped boundary conditions [see, e.g. Chaudhuri and Abu-Arja (1989)], prescribed at all four edges, are given as follows:

$$w = \phi_x = \phi_y = 0. \tag{7}$$

3. BOUNDARY DISCONTINUOUS SOLUTION

A Navier type approach demands that the assumed double Fourier series solution functions satisfy both the governing PDEs and the prescribed boundary conditions *a priori*. Since this would be impossible in the case of rigidly clamped boundary conditions, prescribed at all four edges, either the boundary conditions or the governing PDEs can be satisfied *a priori*. An approach requiring the satisfaction of the boundary conditions *a priori*, if adopted for the problem under investigation, would require that the double Fourier series solution be assumed in the form:

$$\{w, \phi_x, \phi_y\} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{W_{mn}, X_{mn}, Y_{mn}\}\sin(\alpha_m x)\sin(\beta_n y). \tag{8}$$

Substitution of eqn (8) into the system of governing PDEs (6) and equating the coefficients of $\sin(\alpha_m x)\sin(\beta_n y)$, $\sin(\alpha_m x)\cos(\beta_n y)$, etc., in a manner similar to Navier's approach, would yield, in total $9mn$ equations in $3mn$ unknowns, thus failing to provide a solution to this physical problem. This difficulty can be overcome by using a boundary continuous generalized Navier's approach [see Kabir and Chaudhuri (1991)]. An alternative approach, wherein selection of the assumed double Fourier series solution functions depends on the governing PDEs and not on the boundary conditions is presented by Chaudhuri (1989), which is adopted in the present investigation. The underlying mathematical details of the method are available in Chaudhuri (1989), and hence, they are excluded here, in the interest of brevity of presentation. The assumed solution functions for the problem of a rigidly clamped transversely isotropic plate, posed by eqns (6) and (7), are then identical to their simply-supported counterparts, which are given below:

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}\sin(\alpha_m x)\sin(\beta_n y),$$

$$\phi_x(x,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} X_{mn}\cos(\alpha_m x)\sin(\beta_n y),$$

$$\phi_y(x,y) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} Y_{mn}\sin(\alpha_m x)\cos(\beta_n y), \tag{9}$$

with

$$\alpha_m = m\pi/a \quad \text{and} \quad \beta_n = n\pi/b.$$

The total number of unknown constant coefficients introduced in eqns (9) are $3mn+m+n$. The next operation is comprised of differentiation of the assumed solution functions, which is a necessary step before substitution into the equilibrium equations (6). The rigidly clamped boundary condition, prescribed at all four edges, is called the Dirichlet type in the mathematical literature. The procedure for differentiation of the assumed double Fourier series solution functions is based on Lebesgue integration that introduces Fourier coefficients arising from discontinuity of the assumed solution functions at a boundary [see Chaudhuri (1989)]. For the Dirichlet boundary condition under consideration, this procedure alone would introduce more unknown coefficients than equations, thus necessitating the introduction of additional constraints. Since the assumed functions are comprised of sin $(\alpha_m x)$ or cos $(\alpha_m x)$ and sin $(\beta_n y)$ or cos $(\beta_n y)$, the vanishing of displacement or rotation at an edge would automatically lead to the vanishing of the corresponding tangential derivative at that edge and vice versa. Vanishing of the corresponding normal derivative is, however, independent of that of the function itself. Therefore, the following two mutually independent cases of additional constraints are possible, either of which must be introduced in order for the total number of unknowns to become equal to the total number of equations (Chaudhuri, 1989):

Case (i): The displacement functions and their tangential derivatives are not permitted to vanish at an edge. Vanishing of the corresponding normal derivatives at that edge would not constitute a violation of the physics in this case.

Case (ii): The normal derivatives of the displacement functions are not permitted to vanish at an edge. Vanishing of the displacement functions and the corresponding tangential derivatives at that edge would not constitute a violation of the physics in this case.

The present investigation is concerned with the case (i) above, the details of which have been presentd by Chaudhuri (1989) and are omitted here in the interest of brevity of presentation. It is interesting, however, to offer the following physical explanation of the above mathematical operation, as applied to the present problem.

Whenever any of the assumed functions, given by eqns (9), fails to automatically satisfy a prescribed geometric boundary condition, the delinquent solution function is forced to satisfy it. For the case of a rigidly clamped boundary condition, ϕ_x at the edges $x = 0, a$ and ϕ_y at the edges $y = 0, b$ are not satisfied. Therefore, these functions are forced to satisfy the aforementioned boundary conditions, which would yield additional equations to be presented later. Nonetheless, ordinary discontinuities [in the sense of Hobson (1926) and also discussed by Chaudhuri (1989)] may still arise due to the violation of other physical conditions at the boundaries by the assumed solution functions or their first derivatives. They cannot then be further differentiated term-by-term. These functions or their derivatives are then expanded into double Fourier series, in a manner suggested by Chaudhuri (1989), which is similar in form to Green's (1944) and Whitney's (1970) for the case of CPT. This step then paves the way for term-by-term differentiation of the functions or their derivatives, as described below.

The displacement function, w, satisfies the boundary conditions (7a), which implies that its first derivatives can be obtained by termwise differentiation of eqn (9a). Since the two first derivatives do not violate any physical condition at the edges, the three second-derivatives can also be obtained by further termwise differentiation. However, the rotation, ϕ_x, given by eqn (9b) does not satisfy the prescribed boundary conditions at $x = 0, a$. ϕ_x must then be forced to vanish at these two edges, which must satisfy, for all values of $n = 1, 2, 3, \ldots$

$$\sum_{m=1}^{\infty} \delta_m X_{mn} = 0, \quad X_{0n} + \sum_{m=1}^{\infty} \gamma_m X_{mn} = 0, \tag{10a, b}$$

in which

$$(\gamma_m, \delta_m) = \begin{cases} (0,1) & \text{if } m \text{ is odd,} \\ (1,0) & \text{if } m \text{ is even.} \end{cases} \tag{11}$$

Term-by-term differentiation now yields

$$\phi_{x,x} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_m X_{mn} \sin(\alpha_m x)\sin(\beta_n y), \quad 0 < x < a, \quad 0 \leqslant y \leqslant b. \tag{12}$$

A close examination reveals that $\phi_{x,x}$ given by eqn (12), if permitted to vanish at the edges, $x = 0, a$, would imply vanishing curvature at these edges, which would constitute a violation of the physical reality there. This would invalidate term-by-term differentiation with respect to x (i.e. $\phi_{x,xx}$). $\phi_{x,xx}$ must then be obtained by employing the aforementioned procedure due to Green (1944), Whitney (1971) and Chaudhuri (1989) as follows:

$$\phi_{x,xx} = \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin(\beta_n y) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (-\alpha_m^2 X_{mn} + \gamma_m a_n + \delta_m b_n)\cos(\alpha_m x)\sin(\beta_n y), \tag{13}$$

in which constant coefficients a_n and b_n; $n = 1, 2, \ldots,$ are as defined in the Appendix. It may be noted that the above step has generated $2n$ equations (10a, b) and $2n$ constant coefficients (a_n and b_n). The remaining derivatives can be obtained by termwise differentiation.

A similar operation on the rotation function ϕ_y, given by eqn (9c), produces another $2m$ sets of linear algebraic equations arising from boundary conditions, given by

$$\sum_{n=1}^{\infty} \delta_n Y_{mn} = 0, \quad Y_{m0} + \sum_{n=1}^{\infty} \gamma_n Y_{mn} = 0. \tag{14}$$

The derivatives of interest are given by

$$\phi_{y,y} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta_n Y_{mn} \sin(\alpha_m x)\sin(\beta_n y), \quad 0 \leqslant x_1 \leqslant a, \quad 0 < x_2 < b, \tag{15}$$

$$\phi_{y,yy} = \frac{1}{2}\sum_{m=1}^{\infty} c_m \sin(\alpha_m x) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (-\beta_n^2 Y_{mn} + \gamma_n c_m + \delta_n d_m)\sin(\alpha_m x)\cos(\beta_n y). \tag{16}$$

As can be seen from eqn (16), this operation introduces an additional $2m$ constant coefficients, c_m and d_m, presented in eqn (A1c, d), details of which are omitted in the interest of brevity. Therefore, the total number of unknown constant coefficients (W_{mn}, X_{mn}, Y_{mn}, a_n, b_n, c_m, and d_m) are $3mn+3m+3n$, which ask for as many linear algebraic equations to ensure existence of the solution. So far the boundary conditions have supplied $2m+2n$ equations (from satisfying ϕ_x and ϕ_y at the respective edges), given by eqns (10, 14). The remaining $3mn+m+n$ required equations are obtained from satisfying the governing partial differential eqns (6) as follows.

Expansion of the transverse load, q, into double Fourier series

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} \sin(\alpha_m x)\sin(\beta_n y) \tag{17}$$

and substitution of the assumed solution functions and their appropriate derivatives into eqns (6), followed by equating the coefficients of $\sin(\alpha_m x)\cos(\beta_n y)$, etc., supply the following, $3mn+m+n$, sets of linear algebraic equations:

$$\{A[\alpha_m^2 + \beta_n^2]W_{mn} + A[\alpha_m X_{mn} + \beta_n Y_{mn}] + q_{mn}\} = 0, \quad \text{for } m, n = 1, 2, \ldots, \infty, \tag{18a}$$

$$\left\{-A\alpha_m W_{mn} - \left[A + D\alpha_m^2 + \left(\frac{1-\nu}{2}\right)D\beta_n^2\right]X_{mn} - \left(\frac{1+\nu}{2}\right)D\alpha_m\beta_n Y_{mn} + D(\gamma_m a_n + \delta_m b_n)\right\} = 0,$$

$$\text{for } m, n = 1, 2, \ldots, \infty, \tag{18b}$$

$$\left\{-A\beta_n W_{mn} - \left(\frac{1+\nu}{2}\right)\alpha_m\beta_n D X_{mn} - \left[A + \left(\frac{1-\nu}{2}\right)D\alpha_m^2 + D\beta_n^2\right]Y_{mn} + D(\gamma_n c_m + \delta_n d_m)\right\} = 0,$$

$$\text{for } m, n = 1, 2, \ldots, \infty, \tag{18c}$$

$$\left\{\left[A+D\left(\frac{1-\nu}{2}\right)\beta_n^2\right]X_{0n}+\tfrac{1}{2}Da_n\right\}=0,\quad \text{for } n=1,2,\ldots,\infty, \tag{18d}$$

$$\left\{\left[A+D\left(\frac{1-\nu}{2}\right)\alpha_n^2\right]Y_{m0}+\tfrac{1}{2}Dc_m\right\}=0,\quad \text{for } m=1,2,\ldots,\infty. \tag{18e}$$

Equations (18) together with eqns (10), (14) constitute a system of $3mn+3m+3n$ equations in as many unknowns, which must now be solved in a manner that would also ensure computational efficiency, suggested by Chaudhuri (1989). Equations (18) are first solved for W_{mn}, X_{mn} and Y_{mn} in terms of constant coefficients a_n, b_n, etc. These are then substituted in eqn (10), (14) generated from satisfying the geometric boundary conditions. This operation may reduce the size of the problems under consideration by more than an order of magnitude (depending on m, n) finally resulting in $2m+2n$ linear algebraic equations (for each combination of m and n), which can be easily solved.

4. RESULTS AND DISCUSSIONS

The numerical results for moderately thick rectangular plates subjected to uniformly distributed transverse load, q, are presented, with $k^2 = 5/6$. The following normalized quantities are defined:

$$w^* = \frac{Ew^{**}}{30(1-\nu^2)G_t},\quad w^{**} = \frac{10^3Eh^3w}{qa^4},\quad M_x^* = \frac{10^2M_x}{qa^2}. \tag{19}$$

Numerical results on the convergence of w^* and M_x^*, both computed at the center of a moderately-thick ($a/h = 10$) square transversely isotropic plate, are presented in Fig. 2. A reasonably rapid convergence is observed. Accuracy of the numerical results are first ascertained by comparison with their finite element counterparts, computed using the commercial finite element code, NISA-II. For example, w^*, obtained using the present approach (with $m = n = 51$) is 1.42, which compares favorably with $w^* = 1.45$ (approximately 2% difference), computed using NISA-II with 16×16 mesh of quadrilateral 8-node degenerate FSDT-based plate/shell elements (NKTP = 32, order = 2). The accuracy of the present solution is further validated by comparison with the converged solution due to Chaudhuri's (1987) degenerate triangular Mindlin plate bending elements, which testifies

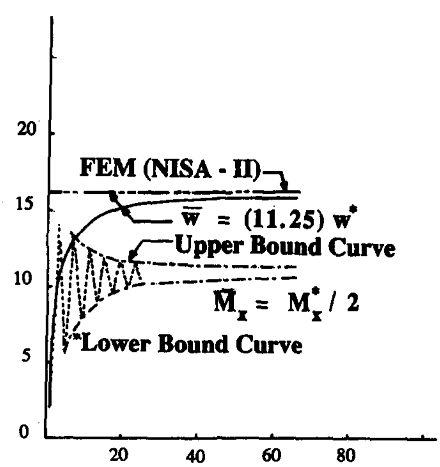

Fig. 2. Convergence of the normalized central deflection and central moment of a square moderately-thick ($a/h = 10$) plate.

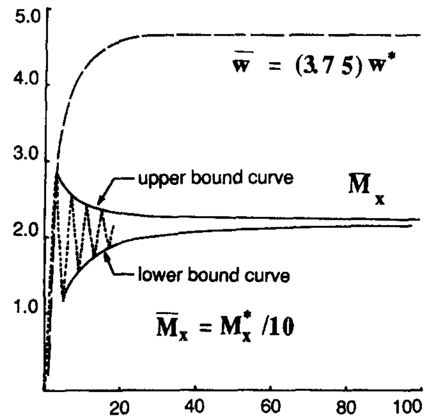

Fig. 3. Convergence of the normalized central deflection and central moment of a square relatively thin ($a/h = 20$) plate.

to the accuracy of all three solutions. Similar results for w^* and M_x^* are also presented for a relatively thinner plate, with $a/h = 20$ (Fig. 3). While w^* displays a resonably rapid monotonic convergence, a bounded oscillatory convergence is observed for M_x^*. The oscillations are shown by a dashed line, while the lower and upper bound curves are represented by chain or solid ones. For sufficiently large m, n, the sum of the Fourier series tends to converge to the average value of upper and lower bounds, which is in accord with the theory of Fourier series, as expounded by Hobson (1926). For example, the error in M_x^*, of a moderately-thick ($a/h = 10$) plate, defined to be {average of lower bound ($m = n = 33$) and upper bound ($m = n = 31$) − average of lower bound ($m = n = 13$) and upper bound ($m = n = 11$)}/{average of lower bound ($m = n = 33$) and upper bound ($m = n = 31$)}, is approximately 2.9%. It may be noted that convergence of central M_x implies its convergence at the edge, because as has been discussed in the preceding section, the Fourier coefficients W_{mn}, X_{mn}, Y_{mn} are expressed in terms of the boundary Fourier coefficients, a_n, b_n, etc., which are related to the first derivatives of rotations at the appropriate edges and hence, edge moments [please see eqns (A2) in the Appendix].

Figure 4 presents the effect of the modular ratio, E/G_t, which is a measure of transverse shear deformability along with thickness, on the central deflection of a transversely isotropic moderately thick ($a/h = 10$) plate, made of pyrolytic graphite material with $\nu = -0.21$. It is noteworthy that the deflection is, for all practical purposes, inversely proportional to the transverse shear modulus, G_t for large E/G_t ratios, which would imply that even a moderately

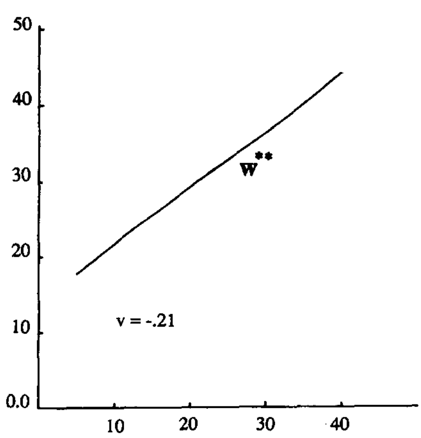

Fig. 4. Variation of the central deflection of a square moderately-thick ($a/h = 10$) pyrolytic graphite plate with respect to the modular ratio, E/G_t.

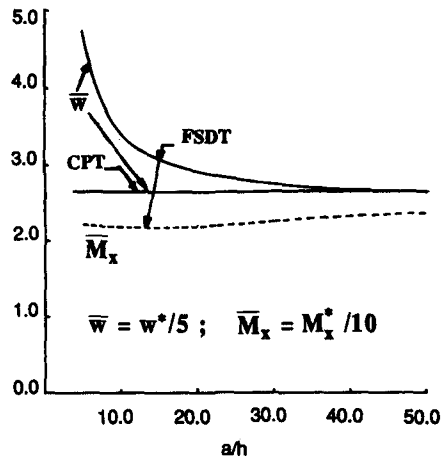

Fig. 5. Variation of the normalized central deflection and moment of a square plate with respect to length to thickness ratio.

thick plate undergoes predominantly large transverse shear deformation and very little flexural deformation, when made of pyrolytic graphite type materials. These results further show that transverse isotropy of pyrolytic graphite plates accentuates the effect of transverse shear deformation by an order of magnitude, and thus constitutes the primary cause of failure in these and the related graphite composite materials (e.g. graphite–epoxy) in the regions of high shear strain [see Chaudhuri (1991)].

Figure 5 presents the variation of the central deflection and moment of a square plate, with respect to length to thickness ratio, a/h. The effect of transverse shear deformation on the computed deflection in the case of thick plates (especially for $a/h < 10$) is self-evident in this plot. It is noteworthy that, for the case of square plates with $a/h = 20$ and 50, the present FSDT-based converged ($m = n = 51$) solutions for central deflection yield $w^* = 1.30$ and 1.21, respectively. The latter compares favorably with the CPT solution, $w^* = 1.19$, computed using the finite difference method as reported by Szilard (1974). The corresponding CPT-based analytical solutions obtained by Green (1944) and Timoshenko and Woinowsky-Krieger (1959), using fewer terms in the series, are 1.27 and 1.26 respectively, while the one-term Galerkin solution, 1.28, is also in the same range of values [see Szilard (1974)]. Figure 5 can be used to obtain approximately (8.5% error) the lower limit (with regard to the length to thickness ratio) of validity of the CPT for a transversely isotropic plate. Influence of the aspect ratio, b/a, on the central displacement and moment of transversely isotropic rectangular plates, with $a/h = 20$, is shown in Fig. 6. Figure 7 displays the variation of deflection and moment along the length of the transversely isotropic

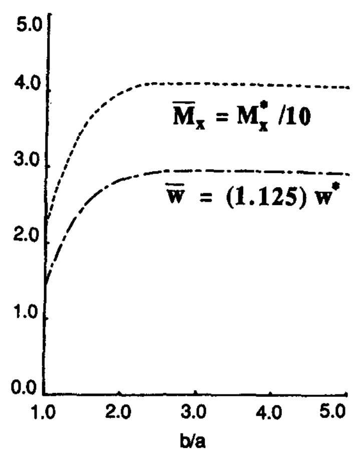

Fig. 6. Variation of the normalized central deflection and moment of a relatively thin ($a/h = 20$) rectangular plate with respect to length to width ratio.

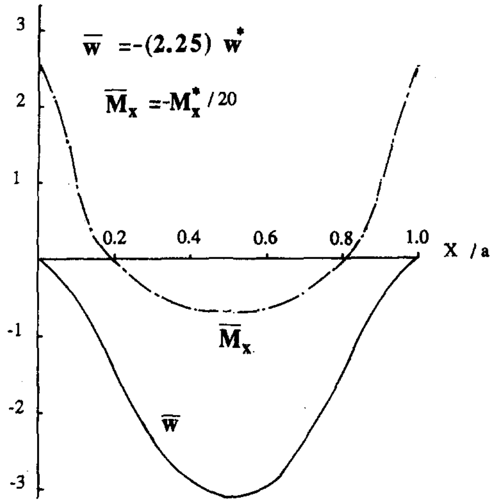

Fig. 7. Variation of deflection and moment along the centerline of a square moderately-thick (a/h = 10) plate.

moderately-thick square plate. These results are identical to their counterparts, obtained by a boundary continuous displacement approach [see Kabir and Chaudhuri (1992)].

5. SUMMARY AND CONCLUSIONS

A heretofore unavailable analytical solution to the boundary-value problem of shear-flexible rectangular rigidly clamped transversely isotropic Mindlin plate is presented. An isotropic plate solution can be recovered as a special case. An accurate yet computationally efficient boundary discontinuous Fourier series technique (BDFST), that has been recently developed, is utilized to solve the three highly coupled second-order partial differential equations with constant coefficients that result from the Mindlin hypothesis. Numerical results presented herein are helpful in (i) validating the accuracy and computational efficiency of the technique presented here, (ii) understanding the nature of convergence of double Fourier series in the presence of edge discontinuities, introduced by the rigidly clamped boundary conditions, (iii) ascertaining the limit of applicability of the classical plate theory, and (iv) providing physical insight into the complex deformation behavior (e.g. thickness effect) of a rigidly clamped moderately thick transversely isotropic (pyrolytic graphite) plate. The numerical results further demonstrate that transverse isotropy (measured by the ratio of in-plane Young's modulus to transverse shear modulus) of pyrolytic graphite plates accentuates the effect of transverse shear deformation by an order of magnitude, and thus constitutes the primary cause of failure in these and the related graphite composite materials (e.g. graphite–epoxy) in the regions of high shear strain. These results are expected to serve as bench-marks for checking the accuracy of various approximate numerical techniques, such as the FEM, in the context of the Mindlin hypothesis.

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APPENDIX

Definitions of constant coefficients, arising from edge discontinuities (presented in Section 3) are given below:

$$a_n = \frac{4}{ab}\int_0^b [\phi_{x,x}(a,y) - \phi_{x,x}(0,y)] \sin(\beta_n y)\,\mathrm{d}y, \tag{A1a}$$

$$b_n = -\frac{4}{ab}\int_0^b [\phi_{x,x}(a,y) + \phi_{x,x}(0,y)] \sin(\beta_n y)\,\mathrm{d}y, \tag{A1b}$$

$$c_m = \frac{4}{ab}\int_0^a [\phi_{y,y}(x,b) - \phi_{y,y}(x,0)] \sin(\alpha_m x)\,\mathrm{d}x, \tag{A1c}$$

$$d_m = -\frac{4}{ab}\int_0^a [\phi_{y,y}(x,b) + \phi_{y,y}(x,0)] \sin(\alpha_m x)\,\mathrm{d}x. \tag{A1d}$$

The first partial derivatives of rotations at certain boundaries may be expressed in terms of the boundary Fourier coefficients as follows:

$$\phi_{1,1}(0,x_2) = -\frac{a}{4}\sum_{n=1}^{\infty}(a_n+b_n)\sin(\beta_n x_2), \tag{A2a}$$

$$\phi_{1,1}(a,x_2) = \frac{a}{4}\sum_{n=1}^{\infty}(a_n-b_n)\sin(\beta_n x_2), \tag{A2b}$$

$$\phi_{2,2}(x_1,0) = -\frac{b}{4}\sum_{m=1}^{\infty}(c_m+d_m)\sin(\alpha_m x_1), \tag{A2c}$$

$$\phi_{2,2}(x_1,b) = \frac{b}{4}\sum_{m=1}^{\infty}(c_m-d_m)\sin(\alpha_m x_1). \tag{A2d}$$